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ON EMBEDDABLE 1-CONVEX SPACES

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1. Introduction

Throughout this paper all complex spaces are assumed to be reduced and with countable topology.

Let X be a complex space. X is said to be *embeddable* if it can be realized as a complex analytic subset of $\mathbb{C}^m \times \mathbb{P}^n$ for some positive integers m and n . For instance, one checks that a complex curve of bounded Zariski dimension is embeddable.

We say that X is *1-convex* if X is a modification at finitely many points of a Stein space Y , i.e., there exist a compact analytic set $S \subset X$ without isolated points and a proper holomorphic map $\pi : X \rightarrow Y$ such that $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$ and π induces an isomorphism between $X \setminus S$ and $Y \setminus \pi(S)$. S is called the *exceptional set* of X and Y the *Remmert's reduction* of X . See [16] for further properties of 1-convex spaces.

A criterion of Schneider [18] says that a 1-convex space X of bounded Zariski dimension is embeddable if, and only if, there is a holomorphic line bundle L over X such that $L|_S$ is ample.

Using this, Bănică [3] proved that a 1-convex complex surface X of bounded Zariski dimension is *embeddable* provided that X does not admit compact two dimensional irreducible components. By extending this Coltoiu ([4], [5]) showed that every connected 1-convex manifold X with 1-dimensional exceptional set is embeddable if $\dim(X) > 3$. This is true also for threefolds X with some exceptions when the exceptional set contains a \mathbb{P}^1 ([5]).

In this short note we reconsider Coltoiu's example from another point of view. This is based on the following proposition which may be of independent interest.

Proposition 1. *Let $Y \subset \mathbb{P}^n$ be a hypersurface of degree d with isolated singularities, $\pi : M \rightarrow Y$ a resolution of singularities, and $H \subset \mathbb{P}^n$ a hyperplane which avoids the singular locus of Y and such that $\Gamma := H \cap Y$ is smooth. Set $X := M \setminus \pi^{-1}(\Gamma)$. Then for $n \geq 4$ the following statements are equivalent:*

- (a) X is embeddable.
- (b) X is Kähler.
- (c) M is projective.

By this and an example due to Moishezon [12] (see also [6]) we obtain:

Theorem 1. *There exists a 1-convex threefold X with exceptional set \mathbb{P}^1 such that X is not Kähler; a fortiori X is not embeddable.*

For the proof of Proposition 1 we use several short exact sequences, Bott's formula, Thom's isomorphism, and some facts on pluriharmonic functions.

Also employing recent results due to Fujiki [9] we prove (see the next section for definitions):

Theorem 2. *Let $\pi : X \longrightarrow Y$ be a finite holomorphic map of complex spaces with X of bounded Zariski dimension. If X is maximal and Y is Hodge, then it holds:*

- (a) *Y compact implies X projective.*
- (b) *Y is 1-convex implies X is 1-convex and embeddable.*

REMARK 1. Note that by [23], 1-convexity is invariant under finite holomorphic surjections. However, this does not hold for embeddability.

As a consequence of Theorem 2 we improve a well-known projectivity criterion due to Grauert [10] to:

Proposition 2. *Let X be a compact complex space. If X is Hodge and maximal, then X is projective.*

and the embeddability result due to Th. Peternell ([17], Theorem 2.6) to:

Proposition 3. *Let X be a 1-convex space of bounded Zariski dimension such that X is Hodge and maximal. Then X is embeddable.*

2. Continuous weakly pluriharmonic functions

Let X be a complex space. As usual, \mathcal{P}_X denotes the sheaf of germs of pluriharmonic functions on X . Then the canonical map $\mathcal{O}_X \longrightarrow \mathcal{P}_X$ given by $f \mapsto \operatorname{Re} f$ induces a short exact sequence

$$(\star) \quad 0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{P}_X \longrightarrow 0.$$

Consider $\widehat{\mathcal{P}}_X :=$ the sheaf of *continuous weakly pluriharmonic functions*, i.e., for every open subset U of X , $\widehat{\mathcal{P}}_X(U)$ consists of those $h \in C^0(U, \mathbb{R})$ which are pluriharmonic on $\operatorname{Reg}(U)$.

Clearly $\mathcal{P}_X \subseteq \widehat{\mathcal{P}}_X$, and if $\widehat{\mathcal{O}}_X$ denotes the sheaf of continuous weakly holomorphic functions, we have a natural map $\widehat{\mathcal{O}}_X \longrightarrow \widehat{\mathcal{P}}_X$ given by $f \mapsto \operatorname{Re} f$.

Here we prove:

Proposition 4. *The canonical short sequence*

$$0 \longrightarrow \mathbb{R} \longrightarrow \widehat{\mathcal{O}}_X \longrightarrow \widehat{\mathcal{P}}_X \longrightarrow 0,$$

is exact.

Proof. We check only the surjectivity of $\widehat{\mathcal{O}}_X \longrightarrow \widehat{\mathcal{P}}_X$. We do this in two steps.

STEP 1. Suppose X is normal. Let $\pi : M \longrightarrow X$ be a resolution of singularities. Then $\pi_* \mathcal{P}_M = \mathcal{P}_X$ by Proposition 2.1 in [9]. Now, since on a complex manifold a continuous real-valued function φ is pluriharmonic if and only if φ and $-\varphi$ are plurisubharmonic we obtain that $\widehat{\mathcal{P}}_X = \mathcal{P}_X$, whence the desired surjectivity in view of (\star) .

STEP 2. The general case. Let $\nu : Y \longrightarrow X$ be the normalization of X . Let $x_o \in X$, U an open neighborhood of x_o , and $h \in \widehat{\mathcal{P}}_X(U)$. Then, by Step 1., $h \circ \nu \in \mathcal{P}_Y(\nu^{-1}(U))$. By Proposition 2.3 in [9] after shrinking $U \ni x_o$, there is $f \in \widehat{\mathcal{O}}_X(U)$ such that $\operatorname{Re} f = h$. Note that in *loc. cit.* this is done under the additional hypothesis $h \in C^\infty(U, \mathbb{R})$. But our case follows *mutatis mutandis*, whence the proposition. \square

Recall ([7], pp. 122–126) that a complex space Z is said to be *maximal* if $\mathcal{O}_Z = \widehat{\mathcal{O}}_Z$ and that every complex space X admits a *maximalization* \widehat{X} , i.e., \widehat{X} is maximal and there is a holomorphic homeomorphism $\pi : \widehat{X} \longrightarrow X$ which induces a biholomorphic map between $\widehat{X} \setminus \pi^{-1}(M(X))$ and $X \setminus M(X)$, where $M(X)$ is the non-maximal locus of X , i.e., $M(X) = \{x \in X; \mathcal{O}_{X,x} \neq \widehat{\mathcal{O}}_{X,x}\}$. Clearly every normal complex space is maximal. For this reason, maximal complex spaces are also called “weakly normal”.

Corollary 1. *If X is maximal, then $\mathcal{P}_X = \widehat{\mathcal{P}}_X$.*

Corollary 2. *If X is normal, then every pluriharmonic function h on $\operatorname{Reg}(X)$ extends uniquely to a pluriharmonic function on X .*

Proof. Since h and $-h$ extend uniquely to plurisubharmonic functions φ and ψ on X , we get $\varphi = -\psi$. Hence φ is continuous, whence φ is pluriharmonic by Corollary 1. \square

By a *d-closed, real (1, 1)-form* (in the sense of Grauert [10]) on a complex space X we mean, a *d-closed, real (1, 1)-form* ω on $\operatorname{Reg}(X)$ such that every point $x \in X$ admits an open neighborhood U on which there is $\varphi \in C^2(U, \mathbb{R})$ with $\omega = i\partial\bar{\partial}\varphi$ on $\operatorname{Reg}(U)$. This φ is called a *local potential function* for ω . We say that ω is *Kähler* if the local potentials may be chosen strongly plurisubharmonic.

Alternatively, by Moishezon [14] we define a *d-closed, real (1, 1)-form* on X as a collection $\{(U_j, \varphi_j)\}_{j \in J}$ where $\{U_j\}_j$ is an open covering of X and $\varphi_j \in C^2(U_j, \mathbb{R})$ are such that $\varphi_j - \varphi_k$ is pluriharmonic. Two such collections $\{(U_j, \varphi_j)\}_{j \in J}$ and $\{(V_k, \psi_k)\}_{k \in K}$ define the same form if $\varphi_j - \psi_k$ is pluriharmonic on $U_j \cap V_k$ for all

indices j and k .

Corollary 3. *For a maximal complex space X the above two notions of d -closed, real $(1, 1)$ -forms coincide in an obvious sense.*

Proof. This is immediate by Corollary 1. \square

To every d -closed, real $(1, 1)$ -form ω on X we associate canonically an element of $H^1(X, \widehat{\mathcal{P}}_X)$, which in turn goes into its *de Rham* class $[\omega] \in H^2(X, \mathbb{R})$ via the cohomology sequence from Proposition 4.

We say that ω is *integral* if its *de Rham* class belongs to $\text{Im}(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{R}))$.

One has the following (see [10], proof of Satz 3)

Lemma 1. *If ω is an integral form on a maximal space X , then there is a holomorphic line bundle $L \rightarrow X$ together with a class C^2 -hermitean metric on L whose Chern form is ω . In particular, if ω is Kähler, then L is positive.*

Let X be a complex space. X is said to be *Kähler* if X has a Kähler form (in the sense of Grauert). We say that X is *Hodge* if it admits a Kähler form which is integral.

Proposition 5. *Let $\pi : Y \rightarrow X$ be a finite holomorphic map of complex spaces such that X is Hodge. Then Y is Hodge. In particular, the maximalization \tilde{X} and the normalization X^* of X are Hodge, too.*

Proof. Let $\{(U_j, \psi_j)\}_j$, $U_j \subseteq X$, defines a Kähler form ω on X . Let $V_j \subseteq U_j$ such that $\{V_j\}_j$ is also a covering of X . Then by [22] for every $\delta \in C^0(X, \mathbb{R})$, $\delta > 0$, there exists $\psi \in C^\infty(Y, \mathbb{R})$, $0 < \psi < \delta$, such that $\sigma_j := \psi_j \circ \pi + \psi$ are strongly plurisubharmonic on $W_j := \pi^{-1}(V_j)$ for all j ; hence $\{(W_j, \sigma_j)\}_j$ defines a Kähler form $\pi^*\omega$ on Y . Of course $\pi^*\omega$ depends on δ and ψ , but this is irrelevant for our discussion. Moreover, in view of a canonical commutative diagram and Proposition 4, if ω is integral, then $\pi^*\omega$ is integral too. \square

Now Lemma 1 and the criteria of Grauert [10] and Schneider [18] give Theorem 2.

REMARK 2. There is a compact, normal, two dimensional complex space X with only one singularity such that $\text{Reg}(X)$ is Kähler, and X is *not* Kähler. (This follows from [14] and [10].)

3. Proof of proposition 1

The only nontrivial implication is (b) \Rightarrow (c) which we now consider. First we state:

CLAIM. The restriction map $H^1(M, \mathcal{P}_M) \longrightarrow H^1(X, \mathcal{P}_M)$ is surjective.

The proof of this will be done in several steps.

STEP 1. *For every abelian group G we have $H^1(\Gamma, G) = 0$.*

Indeed, by a theorem of Siu [19], as $Y \setminus \Gamma$ is a Stein subspace of $\mathbb{P}^n \setminus \Gamma$, it admits a Stein open neighborhood D ; thus $\mathbb{P}^n \setminus \Gamma = D \cup (\mathbb{P}^n \setminus Y)$ is a union of two Stein open subsets. On the other hand, if an n -dimensional complex manifold Ω is a union of q Stein open subsets, then $H_c^i(\Omega, G) = 0$ for $i \leq n - q$. The assertion follows easily.

STEP 2. $H^2(Y, \mathcal{O}_Y) = 0$.

For this, we let \mathcal{I}_Y be the coherent ideal sheaf of Y in \mathbb{P}^n . Then $\mathcal{I}_Y \simeq \mathcal{O}(-[Y])$, where $[Y]$ denotes the canonical line bundle associated to the divisor defined by Y .

Now Bott's formula gives the vanishing of $H^i(\mathbb{P}^n, \mathcal{O}(k))$ for integers i, k with $1 \leq i < n$, and by the long exact cohomology sequence associated to the short exact sequence $0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0$, the assertion of Step 2 results immediately.

STEP 3. *The maps $H^1(M, \mathcal{O}) \longrightarrow H^1(X, \mathcal{O})$ and $H^2(M, \mathcal{O}) \longrightarrow H^2(X, \mathcal{O})$ are surjective and injective respectively.*

Let V be an arbitrary open neighborhood of Γ in Y . Since $Y \setminus \Gamma$ is Stein, the Mayer-Vietoris sequence for $Y = (Y \setminus \Gamma) \cup V$ and Step 2 give that the maps $H^1(V, \mathcal{O}) \longrightarrow H^1(V \setminus \Gamma, \mathcal{O})$ and $H^2(V, \mathcal{O}) \longrightarrow H^2(V \setminus \Gamma, \mathcal{O})$ are surjective and injective respectively.

Assume now $V \subset \text{Reg}(Y)$; hence $\pi^{-1}(V)$ is biholomorphic to V via π . This and the above discussion plus the Mayer-Vietoris sequence for $M = X \cup \pi^{-1}(V)$ completes the proof of Step 3.

STEP 4. $H^2(M, G) \longrightarrow H^2(X, G)$ is surjective for every abelian group G .

We view Γ as a smooth complex hypersurface in M . The inclusion $X \subset M$ gives rise to an exact cohomology sequence (coefficients in any abelian group G)

$$\cdots \longrightarrow H^i(M, X; G) \longrightarrow H^i(M; G) \longrightarrow H^i(X; G) \longrightarrow H^{i+1}(M, X; G) \longrightarrow \cdots$$

On the other hand since Γ is a non-singular complex hypersurface, a tubular neighborhood of Γ is diffeomorphic to a neighborhood of the 0-section of the normal bundle of Γ in M . This bundle being holomorphic is naturally oriented. We thus have, see [2], a Thom isomorphism:

$$H^i(M, X; G) \cong H^{i-2}(\Gamma; G),$$

whence the assertion of Step 4 using Step 1.

(•) The proof of the claim follows by diagram chasing using Steps 3 and 4 and

the next commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^1(M, \mathcal{O}) & \longrightarrow & H^1(M, \mathcal{P}) & \longrightarrow & H^2(M, \mathbb{R}) & \longrightarrow & H^2(M, \mathcal{O}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(X, \mathcal{O}) & \longrightarrow & H^1(X, \mathcal{P}) & \longrightarrow & H^2(X, \mathbb{R}) & \longrightarrow & H^2(X, \mathcal{O}), \end{array}$$

(•) For the proof of the proposition we let $\mathcal{K}_M^{1,1}$ be the sheaf of germs of real smooth $(1, 1)$ -forms on M which are d -closed. As usual, \mathcal{E}_M represents the sheaf of germs of smooth real functions on M . The short exact sequence on M ,

$$0 \longrightarrow \mathcal{P}_M \longrightarrow \mathcal{E}_M \longrightarrow \mathcal{K}_M^{1,1} \longrightarrow 0,$$

where the last non trivial map is given by $\varphi \mapsto \sqrt{-1}\partial\bar{\partial}\varphi$, induces a commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^0(M, \mathcal{E}_M) & \longrightarrow & H^0(M, \mathcal{K}_M^{1,1}) & \longrightarrow & H^1(M, \mathcal{P}_M) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(X, \mathcal{E}_M) & \longrightarrow & H^0(X, \mathcal{K}_M^{1,1}) & \longrightarrow & H^1(X, \mathcal{P}_M) & \longrightarrow & 0. \end{array}$$

By diagram chasing and the above claim if ω is the Kähler form of X , then there are: a smooth, d -closed, real $(1, 1)$ -form α on M and a smooth real-valued function φ on X such that

$$(\P) \quad \alpha|_X - \omega = \sqrt{-1}\partial\bar{\partial}\varphi.$$

Now, select $\chi \in C^\infty(X, \mathbb{R})$ which vanishes on a neighborhood Ω of $\pi^{-1}(\text{Sing}(Y))$ and equals 1 outside a compact subset of X . By (\P) , the smooth $(1, 1)$ -form $\omega + \sqrt{-1}\partial\bar{\partial}(\chi\varphi)$ on X extends trivially to a smooth, real, and d -closed $(1, 1)$ -form $\widehat{\omega}$ on M .

Let β be the canonical Kähler form on \mathbb{P}^n . For every $c > 0$ define a d -closed $(1, 1)$ -form $\widetilde{\omega}_c$ on M by setting:

$$\widetilde{\omega}_c := \widehat{\omega} + c\pi^*(\beta).$$

Clearly $\widetilde{\omega}_c$ restricted to Ω is positive definite for every $c > 0$. On the other hand, there is $c > 0$ sufficiently large such that $\widetilde{\omega}_c$ is positive definite near the compact set $M \setminus \Omega$. Thus M is Kähler. Since M is Moishezon, by [13] M is projective. \square

REMARK 3. In [20] a similar version to our Proposition 1, without any smoothness assumption on $H \cap Y$ and with the additional assumption that $H^2(X, \mathcal{O}_X) = 0$, is stated.

Unfortunately, the “given proof” is wrong. See Colţoiu’s pertinent comments [5] for this and many, many other fatal errors, which, to our unpleasant surprise, are used again in [21].

4. Proof of theorem 1

Let $Y \subset \mathbb{P}^4$ be a hypersurface of degree $d > 2$ having a nondegenerate quadratic point y_o as its only singularity [12]. Let $\sigma : V \longrightarrow \mathbb{P}^4$ be the quadratic transform with center y_o . Set $\Sigma := \sigma^{-1}(y_o)$, $W :=$ the proper transform of Y (W is a nonsingular hypersurface in V), and $T := \Sigma \cap W \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Let S be one of the two factors and $\rho : T \longrightarrow S$ the corresponding projection.

If N denotes the normal bundle of T in W , the restriction of N to each of the fibres of ρ is the negative of the hyperplane bundle, so the criterion of Nakano and Fujiki applies ([8], [15]).

In other words W is obtained by blowing-up a non singular M along a rational non singular curve S . One obtains easily a holomorphic map $\pi : M \longrightarrow Y$ which resolve the singularity y_o of Y and $S = \pi^{-1}(y_o) \simeq \mathbb{P}^1$.

On the other hand, by [6], M is not Kähler if $d > 2$. Therefore, if we choose a linear hyperplane H in \mathbb{P}^4 , $H \not\ni y_o$, such that $H \cap Y$ is smooth, then by Proposition 1, $X := M \setminus \pi^{-1}(Y \cap H)$ is the desired example. \square

REMARK 4. As a counterexample for embeddability this example is due to Colţoiu [5] where by a different method he obtained that $H^1(X, \mathcal{O}_X) = 0$ under the additional hypothesis that H intersects Y transversally.

Here we emphasize the non-Kähler property of the example.

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